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Coefficient Inequalities of Multivalent Analytic Function

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Abstract—*The main aim of the present paper is to obtain a new coefficient inequalities of p-valent functions by introducing and studying some new properties of unified class* $W_N^{\lambda}(\phi,\psi;\eta,\beta,p)$ *involving the Ruschweyh Derivative and Hadamard Products.*

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1. INTRODUCTION AND PRELIMINARIES

Let $A(1)$ denote the class of functions of the form [12,13,14],

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{... (1.1)}$$

Which are analytic in the open unit disk $U = \{z : z \in C \ \& \ |z| < 1\}$. Further, by **S** we shall denote the class of all functions in $A(1)$ which are univalent in U. A function $f(z)$ belonging to $A(1)$ is said to be starlike in U if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \ \forall \ z \in U \qquad (1.2)$$

We denote by **S*** the subclass of $A(1)$ consisting of functions which are starlike in U. Also, a function $f(z)$ belonging to $A(1)$ is said to be convex in U if it satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0 \ \forall \ z \in U \qquad (1.3)$$

We denote by **C** the subclass of $A(1)$ consisting of functions which are convex in U. A function $f(z)$ in $A(1)$ is said to

be close-to-convex of order δ if there exists a function g (z) belonging to **S*** such that

$$\mathrm{Re}\left(\frac{zf'(z)}{g'(z)}\right) > \delta \ \forall \ z \in U \qquad \text{… (1.4)}$$

For some δ (0 ≤ δ < 1), we denote by **K** (δ) the subclass of $A(1)$ consisting of functions which are close-to-convex of order δ in U. It is well known that C ⊂ **S***⊂ **K** (0) ⊂ S

Denote by $A(p)$ the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad \text{…(1.5)}$$

Which are analytic in the punctured (open) disc $U = \{z : z \in C$ and $|z| < 1\}$. some properties of some subclasses of $A(p)$ were studied by Aouf et. al [10]. Denote by $S^*(p,\alpha)$ the class of starlike functions $f \in A(p)$ of order $\alpha (0 \le \alpha < p)$ satisfying

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \ \forall \ z \in U \qquad \text{…(1.6)}$$

Preposition: A function $f \in A(p)$ is said to be convex if $zf' \in S^*(p,\alpha)$ which is known as the *Alexander function property* i.e. $f \in C(p,\alpha) \Leftrightarrow zf' \in S^*(p,\alpha)$.

Now let $C(p,\alpha)$ be the class of convex functions $f \in A(p)$ of order $\alpha(0 \le \alpha < p)$ such that $zf' \in S^*(p,\alpha)$. A function $f \in A(1)$ is said to be in the class of β-uniformly convex functions of order α, denoted by $\beta - UCV(\alpha)$ [4, 5] if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)} - \alpha\right\} \ge \beta\left|\frac{zf''(z)}{f'(z)} - 1\right|, \ \forall \ z \in U$$

and is said to be in a corresponding subclass of $\beta - UCV(\alpha)$, denoted by $\beta - S_p(\alpha)$ if

$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}\geq\beta\left|\frac{zf'(z)}{f(z)}-1\right|$, where $-1\leq\alpha\leq1$ and $z\in U$

The class of uniformly convex and uniformly starlike functions has been extensively studied by Goodman[1,2], Ma and Minda[3]. In fact the class of uniformly β-starlike functions was introduced by Kanas and Wisniowski[9], and for which it can be generalised to $\beta-S_p(\alpha)$, the class of uniformly β-starlike functions of order α.

If f of the form (1.5) and $g(z)=z^p+\sum_{k=1}^{\infty}b_{p+k}z^{p+k}$ are two functions in $A(p)$, then the Convolution of f and g is denoted by $f*g$ and given by

$$(f*g)(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}b_{p+k}z^{p+k}. \quad (1.7)$$

Ruschweyh [7], using the convolution techniques, introduced and studied an important subclass of $A(1)$ the class of prestarlike function of order α, which denoted by $R(\alpha)$. Thus $f\in A(1)$ is said to be prestarlike functions of order $\alpha(0\leq\alpha<1)$ if $f*S_\alpha\in S^*(\alpha)$

Where $S_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}}=z+\sum_{n=2}^{\infty}c_n(\alpha)z^n$ and $c_n(\alpha)=\frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!}$ $n\in N:=\{1,2,3,...\}$). we note that $R(0)=C(0)$ and $R\left(\frac{1}{2}\right)=S^*\left(\frac{1}{2}\right)$. Juneja et. al [8]

Ruschweyh derivative of order α in $A(1)$, which preserves the valences of the function is the convolution of two functions $\frac{z}{(1-z)^{\alpha+1}}$ and $f_1(z)$ where $f_1(z)$ is analytic and univalent in $A(1)$,

Denoted by $D^\alpha(f_1(z))$ and defined by

$D^\alpha(f_1(z))=\frac{z}{(1-z)^{\alpha+1}}*f_1(z)\ \forall\ f_1(z)\in A(1)$ and $\alpha\geq-1$

Now taking $D^{\alpha+1}(f_1(z))=\phi(z)=z+\sum_{n=2}^{\infty}\eta_n z^n$

And $D^\alpha(f_1(z))=\psi(z)=z+\sum_{n=2}^{\infty}\gamma_n z^n\ \forall\ f_1(z)\in A(1)$ and $\alpha\geq-1$

Define the family $S(\phi,\psi,\delta)$ consisting of function $f_1(z)\in A(1)$ so that

$$\mathrm{Re}\left(\frac{f(z)*D^{\alpha+1}(f_1(z))}{f(z)*D^\alpha(f_1(z))}\right)>\delta\ \forall\ z\in U$$

Such that $f(z)*D^\alpha(f_1(z))=f(z)*\psi(z)\neq0$, $\eta_n\geq0$ and $\eta_n\geq\gamma_n\ \forall\ (n\geq2)$.

We let $S(\phi,\psi;\eta,\beta,p)$ [10] denote the set of all functions in $A(p)$ for which

$\mathrm{Re}\left\{1+\frac{1}{\eta}\left(\frac{f(z)*\phi(z)}{f(z)*\psi(z)}-p\right)\right\}>\beta\left|\frac{1}{\eta}\left(\frac{f(z)*\phi(z)}{f(z)*\psi(z)}-p\right)\right|$ where η is positive real number and $\beta\geq0$.

For suitable choices of ϕ, ψ and having $\eta=p-\alpha$, we easily obtain the various subclasses of $A(p)$. For example $S\left(\frac{z}{(1-z)^2},\frac{z}{1-z};p-\alpha,0\right)=S^*(p,\alpha)$, $S\left(\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2};p-\alpha,0\right)=C(p,\alpha)$, $S\left(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}},\frac{z}{(1-z)^{2-2\alpha}};p-\alpha,0\right)=R(p,\alpha)$, $S\left(\frac{z}{(1-z)^2},\frac{z}{1-z};p-\alpha,\beta\right)=\beta-S_p(p,\alpha)$, and Furthermore, note that when $p=1$ we obtain $\mathrm{S}\left(\frac{z}{(1-z)^2},\frac{z}{1-z};1-\alpha,0\right)=S^*(\alpha)$,

$\mathrm{S}\left(\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2};1-\alpha,0\right)=C(\alpha)$, $\mathrm{S}\left(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}},\frac{z}{(1-z)^{2-2\alpha}};1-\alpha,0\right)=R(\alpha)$,

$\mathrm{S}\left(\frac{z}{(1-z)^2},\frac{z}{1-z};1-\alpha,\beta\right)=\beta-S_p(\alpha)$, and $\mathrm{S}\left(\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2};1-\alpha,\beta\right)=\beta-UCV(\alpha)$.

Also denote by $N(p)$ [6] the subclass of $A(p)$ consisting of functions of the form

$$f(z)=z^p-\sum_{k=1}^{\infty}a_{p+k}z^{p+k}. \quad (1.8)$$

Now let us write $S_N(\phi,\psi;\eta,\beta,p)=S(\phi,\psi;\eta,\beta,p)\cap N(p)$ i.e. the class of functions consisting of negative coefficients.

2. COEFFICIENT INEQUALITIES

In this paper we will study the properties of unified presentation of functions $f \in N(p)$ belongs to $W_N^{\lambda}(\phi,\psi;\eta,\beta,p)$ i.e. a unification of subclass of multivalent starlike and subclass of multivalent convex kind of functions. First of all, we state the following result for the purpose of the study.

Lemma 2.1: A function f defined by (1.8) is in the class $S_N(\phi,\psi;\eta,\beta,p)$ if and only if

$$\sum_{k=1}^{\infty}\frac{(1-\beta)\eta_{p+k}-\{p(1-\beta)-\eta\}\gamma_{p+k}}{\eta-(1-\beta)(p-1)}|a_{p+k}|\le 1 \qquad (2.1)$$

Where η is positive real number, $\beta \ge 0$, $\eta_{p+k} \ge 0$, $\gamma_{p+k} \ge 0$ and $\eta_{p+k} \ge \gamma_{p+k}$, $\forall$ k ≥1

Proof**:** now from the definition of the $S_N(\phi,\psi;\eta,\beta,p)$ *we have*

If $f \in S_N(\phi,\psi;\eta,\beta,p) \Rightarrow f \in N(p)$ and

$f \in S(\phi,\psi;\eta,\beta,p)$

$$\Rightarrow f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k}. \qquad (2.1.1)$$

Also from $f \in S(\phi,\psi;\eta,\beta,p)$

it satisfies the condition

$$\mathrm{Re}\left\{1+\frac{1}{\eta}\left(\frac{f(z)*\phi(z)}{f(z)*\psi*(z)}-p\right)\right\} > \beta\left|\frac{1}{\eta}\left(\frac{f(z)*\phi(z)}{f(z)*\psi(z)}-p\right)\right| \qquad (2.1.2)$$

Where η is positive real number and $\beta \ge 0$.

Where

$$\phi(z) = D^{\alpha+1}(f_p(z)) = \frac{z^p}{(1-z^p)^{\alpha+2}} * f_p(z) = z^p + \sum_{k=1}^{\infty}\eta_{k+p} z^{k+p}$$

And

$$\psi(z) = D^{\alpha}(f_p(z)) = \frac{z^p}{(1-z)^{\alpha+1}} * f_p(z) = z^p + \sum_{k=1}^{\infty}\gamma_{k+p} z^{k+p} \quad \forall$$

$f_p(z) \in A(p)$

s.t $\eta_{p+k} \ge 0$, $\gamma_{p+k} \ge 0$ and $\eta_{p+k} \ge \gamma_{p+k}$, $\forall$ k ≥1

$$\beta\left|\frac{1}{\eta}\left(\frac{f(z)*\phi(z)}{f(z)*\psi(z)}-p\right)\right| - \mathrm{Re}\left\{\frac{1}{\eta}\left(\frac{f(z)*\phi(z)}{f(z)*\psi(z)}-p\right)\right\} < 1$$

Now we know Re(x) ≤ |x| and |x-y| ≥ |x|-|y| so

$$\frac{\beta}{\eta}\left|\left(\frac{z^p-\sum_{k=1}^{\infty}a_{p+k}\eta_{p+k}z^{p+k}}{z^p-\sum_{k=1}^{\infty}a_{p+k}\gamma_{p+k}z^{p+k}}-p\right)\right| - \left\{\frac{1}{\eta}\left|\left(\frac{z^p-\sum_{k=1}^{\infty}a_{p+k}\eta_{p+k}z^{p+k}}{z^p-\sum_{k=1}^{\infty}a_{p+k}\gamma_{p+k}z^{p+K}}-p\right)\right|\right\} \le 1$$

Now $f(z)$, $\phi(z)$ & $\psi(z)$ are analytic in unit disk ,so

$$\frac{\beta}{\eta}\left(\frac{1-\sum_{k=1}^{\infty}|a_{p+k}|\eta_{p+k}}{1-\sum_{k=1}^{\infty}|a_{p+k}|\gamma_{p+k}}-p\right) - \left\{\frac{1}{\eta}\left(\frac{1-\sum_{k=1}^{\infty}|a_{p+k}|\eta_{p+k}}{1-\sum_{k=1}^{\infty}|a_{p+k}|\gamma_{p+k}}-p\right)\right\} \le 1$$

$$\left(\frac{\beta}{\eta}-\frac{1}{\eta}\right)\left(\frac{1-\sum_{k=1}^{\infty}|a_{p+k}|\eta_{p+k}}{1-\sum_{k=1}^{\infty}|a_{p+k}|\gamma_{p+k}}-p\right) \le 1$$ Since $\beta \ge 0$,

$\eta_{p+k} \ge 0$, $\gamma_{p+k} \ge 0$

$$(\beta-1)\left(\left(1-\sum_{k=1}^{\infty}a_{p+k}|\eta_{p+k}-p\left(1-\sum_{k=1}^{\infty}a_{p+k}|\gamma_{p+k}\right)\right)\right) \le \eta\left(1-\sum_{k=1}^{\infty}a_{p+k}|\gamma_{p+k}\right)$$

$$\sum_{k=1}^{\infty}\frac{(1-\beta)\eta_{p+k}-\{p(1-\beta)-\eta\}\gamma_{p+k}}{\eta-(1-\beta)(p-1)}|a_{p+k}|\le 1$$

Inequality holds.

Where $\beta \ge 0$, $\eta_{p+k} \ge 0$, $\gamma_{p+k} \ge 0$ and $\eta_{p+k} \ge \gamma_{p+k}$, $\forall$ k ≥1

Conversely,

If given inequality (2.1) hold then by proceeding above in reverse order then we get $f \in S_N(\phi,\psi;\eta,\beta,p)$, proves the lemma.

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